it." Sometimes, in fact, seeing a creative solution can be inhibiting, for even though we admire it, we may not think that we could ever do it on our own. While it is true that some people do seem to be naturally more creative than others, we believe that almost everyone can learn to become more creative. Part of this process comes from cultivating a confident attitude, so that when you see a beautiful solution, you no longer think, "I could never have thought of that," but instead think, "Nice idea! It's similar to ones I've had. Let's put it to work!"

Learn to shamelessly appropriate new ideas and make them your own.

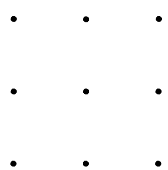
There's nothing wrong with that; the ideas are not patented. If they are beautiful ideas, you should excitedly master them and use them as often as you can, and try to stretch them to the limit by applying them in novel ways. Always be on the lookout for new ideas. Each new problem that you encounter should be analyzed for its "novel idea" content. The more you get used to appropriating and manipulating ideas, the more you will be able to come up with new ideas of your own.

One way to heighten your receptiveness to new ideas is to stay "loose," to cultivate a sort of mental **peripheral vision**. The receptor cells in the human retina are most densely packed near the center, but the most sensitive receptors are located on the periphery. This means that on a bright day, whatever you gaze at you can see very well. However, if it is dark, you will not be able to see things that you gaze at directly, but you will perceive, albeit fuzzily, objects on the periphery of your visual field (try Exercise 2.1.10). Likewise, when you begin a problem solving investigation, you are "in the dark." Gazing directly at things won't help. You need to relax your vision and get ideas from the periphery. Like Pólya's mouse, constantly be on the lookout for twists and turns and tricks. Don't get locked into one method. Try to consciously break or bend the rules.

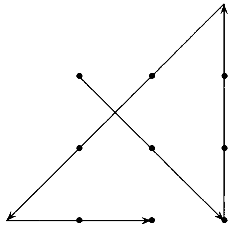
Here are a few simple examples, many of which are old classics. As always, don't jump immediately to the solution. Try to solve or at least think about each problem first!

Now is a good time to fold a sheet of paper in half or get a large index card to hide solutions so that you don't succumb to temptation and read them before you have thought about the problems!

Example 2.1.3 Connect all nine points below with an unbroken path of four straight lines.



Solution: This problem is impossible unless you liberate yourself from the artificial boundary of the nine points. Once you decide to draw lines that extend past this boundary, it is pretty easy. Let the first line join three points, and make sure that each new line connects two more points.



■

Example 2.1.4 Pat wants to take a 1.5-meter-long sword onto a train, but the conductor won't allow it as carry-on luggage. And the baggage person won't take any item whose greatest dimension exceeds 1 meter. What should Pat do?

Solution: This is unsolvable if we limit ourselves to two-dimensional space. Once liberated from Flatland, we get a nice solution: The sword fits into a $1 \times 1 \times 1$-meter *box*, with a long diagonal of $\sqrt{1^2+1^2+1^2} = \sqrt{3} > 1.69$ meters. ■

Example 2.1.5 What is the next letter in the sequence O, T, T, F, F, S, S, E . . . ?

Solution: The sequence is a list of the first letters of the numerals one, two, three, four, . . . ; the answer is "N," for "nine." ■

Example 2.1.6 Fill in the next column of the table.

1	3	9	3	11	18	13	19	27	55	
2	6	2	7	15	8	17	24	34	29	
3	1	5	12	5	13	21	21	23	30	

Solution: Trying to figure out this table one row at a time is pretty maddening. The values increase, decrease, repeat, etc., with no apparent pattern. But who said that the patterns had to be in rows? If you use peripheral vision to scan the table *as a whole* you will notice some familiar numbers. For example, there are lots of multiples of three. In fact, the first few multiples of three, in order, are hidden in the table.

1	3	**9**	3	11	**18**	13	19	**27**	55	
2	**6**	2	7	**15**	8	17	**24**	34	29	
3	1	5	**12**	5	13	**21**	21	23	**30**	

And once we see that the patterns are diagonal, it is easy to spot another sequence, the primes!

1	*3*	**9**	3	*11*	**18**	13	*19*	**27**	55	
2	**6**	2	*7*	**15**	8	*17*	**24**	34	*29*	
3	1	*5*	**12**	5	*13*	**21**	21	*23*	**30**	

The sequence that is left over is, of course, the Fibonacci numbers (Problem 1.3.18). So the next column of the table is 31, 33, 89. ■

Example 2.1.7 Find the next member in this sequence.[2]

$$1, 11, 21, 1211, 111221, \ldots$$

Solution: If you interpret the elements of the sequence as numerical quantities, there seems to be no obvious pattern. But who said that they are numbers? If you look at the relationship between an element and its predecessor and focus on "symbolic" content, we see a pattern. Each element "describes" the previous one. For example, the third element is 21, which can be described as "one 2 and one 1," i.e., 1211, which is the fourth element. This can be described as "one 1, one 2 and two 1s," i.e., 111221. So the next member is 312211 ("three 1s, two 2s and one 1"). ■

Example 2.1.8 Three women check into a motel room that advertises a rate of \$27 per night. They each give \$10 to the porter, and ask her to bring back three dollar bills. The porter returns to the desk, where she learns that the room is actually only \$25 per night. She gives \$25 to the motel desk clerk, returns to the room, and gives the guests back each one dollar, deciding not to tell them about the actual rate. Thus the porter has pocketed \$2, while each guest has spent $10 - 1 =$ \$9, a total of $2 + 3 \times 9 =$ \$29. What happened to the other dollar?

Solution: This problem is deliberately trying to mislead the reader into thinking that the profit that the porter makes plus the amount that the guests spend *should* add up to \$30. For example, try stretching things a bit: what if the actual room rate had been \$0? Then the porter would pocket \$27 and the guests would spend \$27, which adds up to \$54! The actual "invariant" here is not \$30, but \$27, the amount that the guests spend, and this will always equal the amount that the porter took (\$2) plus the amount that went to the desk (\$25). ■

Each example had a common theme: Don't let self-imposed, unnecessary restrictions limit your thinking. Whenever you encounter a problem, it is worth spending a minute (or more) asking the question, "Am I imposing rules that I don't need to? Can I change or **bend the rules** to my advantage?"

Nice guys may or may not finish last, but

> *Good, obedient boys and girls solve fewer problems than naughty and mischievous ones.*

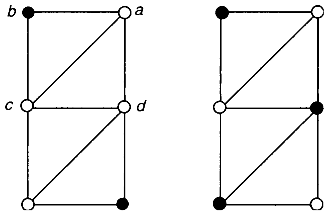
Break or at least bend a few rules. It won't do anyone any harm, you'll have fun, and you'll start solving new problems.

We conclude this section with the lovely "Affirmative Action Problem," originally posed (in a different form) by Donald Newman. While mathematically more sophisti-

[2] We thank Derek Vadala for bringing this problem to our attention. It appears in [42], p. 277.

cated than the monk problem, it too possesses a very brief and imaginative "one-liner" solution. The solution that we present is due to Jim Propp.

Example 2.1.9 Consider a network of finitely many balls, some of which are joined to one another by wires. We shall color the balls black and white, and call a network "integrated" if each white ball has at least as many black as white neighbors, and vice versa. The example below shows two different colorings of the same network. The one on the left is not integrated, because ball a has two white neighbors (c,d) and only one black neighbor (b). The network on the right is integrated.



Given any network, is there a coloration that integrates it?

Solution: The answer is "yes." Let us call a wire "balanced" if it connects two differently colored balls. For example, the wire connecting a and b in the first network shown above is balanced, while the wire connecting a and c is not. Then our one-line solution is to

Maximize the balanced wires!

Now we need to explain our clever solution! Consider all the possible different colorings of a given network. There are finitely many colorings, so there must be one coloring (perhaps more than one) that produces the maximal number of balanced wires. We claim that *this* coloring is integrated. Assume, on the contrary, that it is not integrated. Then, there must be some ball, call it A, colored (without loss of generality) white, that has more white neighbors than black neighbors. Look at the wires emanating from A. The only balanced wires are the ones that connect A with black balls. More wires emanating from A are unbalanced than balanced. However, if we recolored A black, then more of the wires would be balanced rather than unbalanced. Since recoloring A affects only the wires that emanate from A, we have shown that recoloring A results in a coloration with more balanced wires than before. *That contradicts our assumption that our coloring already maximized the number of balanced wires!*

To recap, we showed that if a coloring is not integrated, then it cannot maximize balanced wires. Thus a coloring that maximizes balanced wires must be integrated! ∎

What are the novel ideas in this solution? That depends on how experienced you are, of course, but we can certainly isolate the stunning crux move: the idea of maximizing the number of balanced wires. The underlying idea, the **extreme principle**, is actually a popular "folklore" tactic used by experienced problem solvers (see Section 3.2 below). At first, seeing the extreme principle in action is like watching a karate expert break a board with seemingly effortless power. But once you master it for your own use, you will discover that breaking at least some boards isn't all that difficult. Another notable feature of this solution was the skillful use of argument by contradiction. Again, this is a fairly standard method of proof (see Section 2.3 below).

This doesn't mean that Jim Propp's solution wasn't clever. Indeed, it is one of the neatest one-liner arguments we've ever seen. But part of its charm is the simplicity of its ingredients, like origami, where a mere square of paper metamorphoses into surprising and beautiful shapes. Remember that the title of this book is *The Art and Craft of Problem Solving*. Craft goes a long way, and this is the route we emphasize, for without first developing craft, good art cannot happen. However, ultimately, the problem solving experience is an aesthetic one, as the Affirmative Action problem shows. The most interesting problems are often the most beautiful; their solutions are as pleasing as a good poem or painting.

OK, back to Earth! How do you become a board-breaking, paper-folding, arts-and-crafts Master of Problem Solving? The answer is simple:

Toughen up, loosen up, and practice.

Toughen up by gradually increasing the amount and difficulty of your problem solving work. **Loosen up** by deliberately breaking rules and consciously opening yourself to new ideas (including shamelessly appropriating them!). Don't be afraid to play around, and try not to let failure inhibit you. Like Pólya's mouse, several failed attempts are perfectly fine, as long as you keep trying other approaches. And unlike Pólya's mouse, you won't die if you don't solve the problem. It's important to remember that. Problem solving isn't easy, but it should be fun, at least most of the time!

Finally, **practice** by working on lots and lots and lots of problems. Solving them is not as important. It is very healthy to have several unsolved problems banging around your conscious and unconscious mind. Here are a few to get you started.

Problems and Exercises

The first few (2.1.10–2.1.12) are mental training exercises. You needn't do them all, but please read each one, and work on a few (some of them require ongoing expenditures of time and energy, and you may consider keeping a journal to help you keep track). The remainder of the problems are mostly brain teasers, designed to loosen you up, mixed with a few open-ended questions to fire up your backburners.

2.1.10 Here are two fun experiments that you can do to see that your peripheral vision is both less acute yet more sensitive than your central vision.

1. On a clear night, gaze at the Pleiades constellation, which is also called the Seven Sisters because it has seven prominent stars. Instead of looking directly at the constellation, try glancing at the Pleiades with your peripheral vision; i.e., try to "notice" it, while not quite looking at it. You should be able to see more stars!
2. Stare straight ahead at a wall while a friend slowly moves a card with a letter written on it into the periphery of your visual field. You will notice the movement of the card long before you can read the letter on it.

2.1.11 Many athletes benefit from "cross-training," the practice of working out regularly in another sport in order to enhance performance in the target sport. For example, bicycle racers may lift weights or jog. While we advocate devoting most of your energy to *math* problems, it may be helpful to diversify. Here a few suggestions.