If a is a non-negative real number, $\sqrt{a}$ denotes the principal square root of a. Recall that $\sqrt{0} = 0$ and

if $a > 0$, then $\sqrt{a}$ is the positive square root of a. It follows easily that if $a \geq 0$ and $b \geq 0$, then

$\sqrt{ab} = \sqrt{a}\,\sqrt{b}$.

Note that the above definition of $\sqrt{a}$ is applicable only to non-negative real numbers a.

In order to answer the question posed, we first need to understand the meaning given to $\sqrt{a}$ when a is a negative real number and, more generally, when a is a complex number. In other words, we need a definition for $\sqrt{a}$, the principal square root of a, when a is a complex number.

Since we know that $\sqrt{0} = 0$, let $z \in \mathbb{C}$ and $z \neq 0$.

First, write z in the polar form: $z = r\,(\cos\theta + i\sin\theta)$,

where $r = |z|\ (> 0)$ and $\theta = \operatorname{Arg} z\ \ (-\pi < \theta \leq \pi)$, the principal argument of z.

We shall define $\sqrt{z}$, the principal square root of z, by

$$\sqrt{z} = \sqrt{r}\,\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right).$$

Note that this agrees with $\sqrt{-1} = i$, since $-1 = 1(\cos\pi + i\sin\pi)$, we have

$$\sqrt{-1} = \sqrt{1}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = i.$$

Now, let $a, b \in \mathbb{C}$ with $a \neq 0$ and $b \neq 0$.

Also, let $|a| = r_1$, $|b| = r_2$, $\operatorname{Arg}(a) = \theta_1$ and $\operatorname{Arg}(b) = \theta_2$.

Then $a = r_1(\cos\theta_1 + i\sin\theta_1)$ and $b = r_2(\cos\theta_2 + i\sin\theta_2)$, where $r_1, r_2 > 0$ and $-\pi < \theta_1, \theta_2 \leq \pi$.

By definition, $\sqrt{a} = \sqrt{r_1}\left(\cos\frac{\theta_1}{2} + i\sin\frac{\theta_1}{2}\right)$ and $\sqrt{b} = \sqrt{r_2}(\cos\frac{\theta_1}{2} + i\sin\frac{\theta_1}{2})$.

$\therefore\ \sqrt{a}\,\sqrt{b} = \sqrt{r_1 r_2}\left[\cos\frac{(\theta_1+\theta_2)}{2} + i\sin\frac{(\theta_1+\theta_2)}{2}\right]$ ---------- (1)

Next, in order to find $\sqrt{ab}$, we need the principal argument of ab.

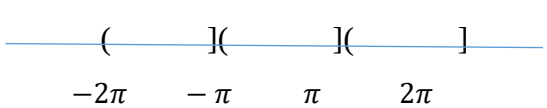
We know that $|ab| = r_1 r_2\ (> 0)$.

Note that $ab = r_1(\cos\theta_1 + i\sin\theta_1)\,.\,r_2(\cos\theta_2 + i\sin\theta_2)$

$$= r_1\, r_2[\cos(\theta_1 + \theta_2) + i\,\sin(\theta_1 + \theta_2)].$$

Since $-\pi < \theta_1,\ \theta_2 \leq \pi$, we have $-2\pi < \theta_1 + \theta_2 \leq 2\pi$.

$\therefore$ Arg (ab) splits into three cases:



Case (i) $\boxed{-\pi < \theta_1 + \theta_2 \leq \pi}$

In this case, $\text{Arg}\,(ab) = \theta_1 + \theta_2$.

$\therefore \sqrt{ab} = \sqrt{r_1 r_2}\,[\cos\frac{(\theta_1+\theta_2)}{2} + i\sin\frac{(\theta_1+\theta_2)}{2}]$ ---------- (2)

Now, (1) and (2) $\Rightarrow$ $\sqrt{ab} = \sqrt{a}\,\sqrt{b}$.

Case (ii) $\boxed{-2\pi < \theta_1 + \theta_2 \leq -\pi}$

This is equivalent to $0 < \theta_1 + \theta_2 + 2\pi \leq \pi$

$\therefore$ $\text{Arg}\,(ab) = \theta_1 + \theta_2 + 2\pi$.

Hence. $\sqrt{ab} = \sqrt{r_1 r_2}\,[\cos\frac{(\theta_1+\theta_2+2\pi)}{2} + i\sin\frac{(\theta_1+\theta_2+2\pi)}{2}]$

$= \sqrt{r_1 r_2}\,[\cos(\frac{(\theta_1+\theta_2}{2} + \pi) + i\sin(\frac{(\theta_1+\theta_2}{2} + \pi)]$

$= -\sqrt{r_1 r_2}\,[\cos(\frac{(\theta_1+\theta_2}{2}) + i\sin(\frac{(\theta_1+\theta_2}{2})]$ ---------- (3)

Now, (1) and (3) $\Rightarrow$ $\sqrt{ab} = -\sqrt{a}\,\sqrt{b}$.

Case (iii) $\boxed{\pi < \theta_1 + \theta_2 \leq 2\pi}$

This is equivalent to $-\pi < \theta_1 + \theta_2 - 2\pi \leq 0$.

$\therefore$ $\text{Arg}\,(ab) = \theta_1 + \theta_2 - 2\pi$.

As in Case (ii), we see that $\sqrt{ab} = -\sqrt{a}\,\sqrt{b}$.

<u>Conclusion:</u>

let $a, b \in \mathbb{C}$ with $ab \neq 0$.

Then, $\sqrt{ab} = \sqrt{a}\,.\sqrt{b} \quad \Leftrightarrow \; -\pi < Arg\,(a) + Arg\,(b) \leq \pi$.

Obviously, if $a, b \in \mathbb{C}$ with $ab = 0$, then $\sqrt{ab} = \sqrt{a}\,.\sqrt{b}$.

<u>Remark:</u>

$\sqrt{(-1)(-1)} \neq \sqrt{-1}\,.\sqrt{-1}$.

By case (iii),

$(-1)(-1) = -\sqrt{-1}\,.\sqrt{-1}$.