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Author(s): Loren C. Larson
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A Discrete Look at $1 + 2 + \cdots + n$

Loren C. Larson

Loren Larson is Professor of Mathematics at St. Olaf College, where he has been a member of the faculty since 1963. He received his Ph.D. in mathematics from the University of Kansas and has done postdoctoral study at Stanford University and at the University of Minnesota. His professional interests are algebra, logic, and combinatorics. An avid problem solver, Dr. Larson is a frequent contributor to problem sections in various mathematics journals. For the past ten years, he has prepared the Mathematical Olympiad and Putnam problem solutions for Mathematics Magazine, and currently is serving as codirector of the Putnam Mathematics Examination. He is the author of two books, Algebra and Trignonometry Refresher and Problem-Solving Through Problems. Professor Larson was the director of a St. Olaf College Mathematics Department project funded by the Sloan Foundation to explore ways in which methods of discrete mathematics can be introduced into the freshman and sophomore curriculum.

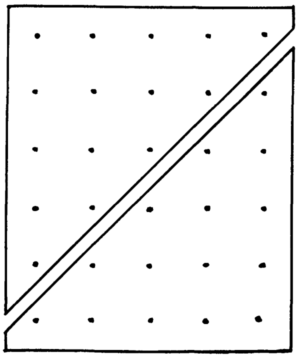
The mathematics of computer science can be distinguished from the mathematics of the physical sciences. A serious study of physics and chemistry usually requires a background in continuous mathematics: calculus, differential equations, real and complex analysis. On the other hand, a study of computer science is more likely to require a knowledge of discrete mathematics: logic, combinatorics, graph theory, operations research, and applied algebra. Both continuous and discrete mathematics grow out of a need to solve real-world problems, and both impart ideas that are ultimately applicable. Both provide a context for developing general problem-solving skills. Ideally, students should have a background in both areas since they overlap and parallel each other in many ways.

The purpose of this article is to look at some problem-solving techniques that are typically encountered in a first course in discrete mathematics. In order to unify this survey, we will show how each idea can be used to prove the well-known formula

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

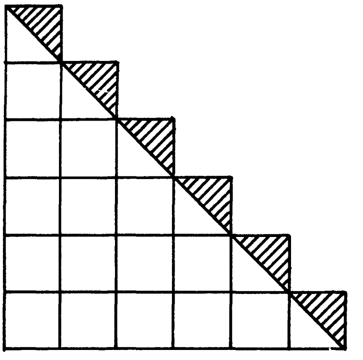
Each section will conclude with exercises which the reader can use to reinforce a feeling for the method.

1. Draw a Figure. A time-honored first step in understanding a problem is to draw and label a figure, a diagram, or a graph as, for example, in Figure 1 and Figure 2 (due to Ian Richards).



$2(1+2+\cdots+n)=n(n+1)$

Figure 1.



$1+2+\cdots+n=(\frac{1}{2})(n^2)+(\frac{1}{2})(n)$

Figure 2.

In discrete mathematics, it is also often useful to represent objects by points, and relationships between objects by edges adjoining corresponding points. Figure 3, for example, represents the flow of water from a source S, through aqueducts, to a terminal point T, at the rates indicated by the weights on the edges. Since the incoming and outgoing rates are the same at each pumping station (i.e., juncture point) along the way, the total output at S must equal the total input at T; that is,

$$(5)(6)=2+4+6+8+10,$$

or equivalently,

$$1+2+3+4+5=\frac{(5)(6)}{2}.$$

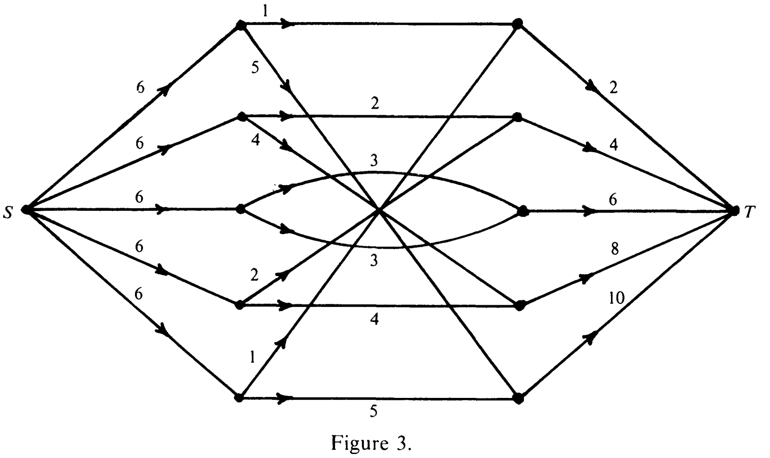


Figure 3.

A similar figure in the general case yields the familiar formula,

$$1+2+\cdots+n=\frac{n(n+1)}{2}.$$

Exercise. Construct a network, as in the preceding example, to show that

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

Exercise. The diagram in Figure 4 is made up of two 1×1 squares, three 2×2 squares, four 3×3 squares, five 4×4 squares, and six 5×5 squares.

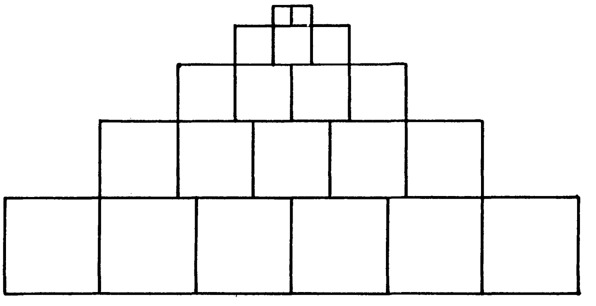


Figure 4.

Show how to extend this figure to establish the general formula

$$2(1 + 2 + \cdots + n) = n(n + 1).$$

Exercise. Find the sum $1 + 2 + \cdots + n$ by an appropriate application of

(a) Pick's Theorem. *The area of a simple lattice polygon (a polygon with lattice points as vertices whose sides do not cross) is given by* $I + (1/2)B - 1$, *where* I *and* B *denote, respectively, the number of interior and boundary lattice points of the polygon.*

(b) Euler's Theorem (on connected planar graphs). $V - E + F = 2$, *where* V, E, F *are, respectively, the number of vertices, edges, and faces in the graph.*

2. Search for a Pattern. The process of algorithm design usually begins with a look at certain low-order special cases, with the hope of discovering patterns that might generalize to higher order, more complicated, cases. The "search for patterns" is a common theme in discrete mathematics. For example, a first step in discovering a closed-formula expression for the sum S_n of the first n positive integers is to compute S_n for $n = 1, 2, 3, 4, 5$. Thus,

$$\begin{aligned} S_1 &= 1 &&= 1 \\ S_2 &= 1 + 2 &&= 3 \\ S_3 &= 1 + 2 + 3 &&= 6 \\ S_4 &= 1 + 2 + 3 + 4 &&= 10 \\ S_5 &= 1 + 2 + 3 + 4 + 5 &&= 15. \end{aligned}$$

From this beginning, it is not difficult to conjecture that the general formula is $S_n = n(n + 1)/2$. (A proof based on this beginning is an easy application of mathematical induction; see Section 3.)

The binomial coefficients provide a rich context for mathematical discovery. One pattern that is relevant here is that the sum of the first k numbers in a "falling diagonal" in Pascal's triangle is equal to the kth number in the next falling diagonal. Thus, (Figure 5) $1 + 4 + 10 + 20 + 35 = 70$.

$$
\begin{array}{ccccccccccccccccc}
 & & & & & & d_0 \searrow & & 1 & & & & & & & & \\
 & & & & & d_1 \searrow & & 1 & & 1 & & & & & & & \\
 & & & & d_2 \searrow & & 1 & & 2 & & 1 & & & & & & \\
 & & & d_3 \searrow & & 1 & & 3 & & 3 & & 1 & & & & & \\
 & & \cdot^{\cdot^{\cdot}} & & 1 & & 4 & & 6 & & 4 & & 1 & & & & \\
 & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & \\
 & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & \\
 & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & \\
1 & & 8 & & 28 & & 56 & & 70 & & 56 & & 28 & & 8 & & 1
\end{array}
$$

Figure 5. Illustration of the "hockey stick" formula.

In terms of binomial coefficients, Figure 5 illustrates that

$$\binom{3}{0} + \binom{4}{1} + \binom{5}{2} + \binom{6}{3} + \binom{7}{4} = \binom{8}{4}.$$

In general,

$$\binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \cdots + \binom{n+k}{k} = \binom{n+k+1}{k}. \qquad (*)$$

This is proved by repeated use of the addition formula:

$$
\begin{aligned}
\binom{n+k+1}{k} &= \binom{n+k}{k} + \binom{n+k}{k-1} \\
&= \binom{n+k}{k} + \left[\binom{n+k-1}{k-1} + \binom{n+k-1}{k-2}\right] \\
&= \binom{n+k}{k} + \binom{n+k-1}{k-1} + \left[\binom{n+k-2}{k-2} + \binom{n+k-2}{k-3}\right] \\
&\;\;\vdots \\
&= \binom{n+k}{k} + \binom{n+k-1}{k-1} + \cdots + \binom{n}{0},
\end{aligned}
$$

since $\binom{n}{i} = 0$ for $i < 0$.

Diagonal d_1 has entries $1, 2, 3, \ldots$. Therefore, taking $n = 1$ and letting $k = n - 1$ in $(*)$, the discovery above shows that

$$1 + 2 + \cdots + n = \binom{1}{0} + \binom{2}{1} + \cdots + \binom{n}{n-1} = \binom{n+1}{n-1} = \frac{(n+1)n}{2}.$$

Exercise. Let r_n denote the nth row of Pascal's triangle (note that r_n has $n + 1$ elements). By examining the first few cases, find formulas for

(a) the sum of every number in r_n;

(b) the sum of every other number in r_n;

(c) the sum of every third number in r_n. [Note: The answer to (c) depends upon the value of n (mod 6).]

3. Mathematical Induction. Mathematical induction is the most important proof technique in discrete mathematics. It is the principal way of proving that a proposed algorithm will, in fact, always produce the desired output for an arbitrary input.

A key step in formulating a proof by mathematical induction is to choose the proper inductive assumption. As an example, consider the following problem:

The coefficient of x^2 *in the expansion of* $(1 + x)^{n+1}$ *is* $1 + 2 + \cdots + n$.

Suppose we let $P(n)$ be the statement that the coefficient of x^2 in $(1 + x)^{n+1}$ is $1 + 2 + \cdots + n$. The result is true when $n = 1$. Assume the result is true when $n = k$. Then $(1 + x)^{k+2} = (1 + x)(1 + x)^{k+1}$ and, by induction, the x^2 coefficient of $(1 + x)^{k+1}$ is $1 + 2 + \cdots + k$; that is,

$$(1 + x)^{k+2} = (1 + x)\left(a_0 + a_1 x + (1 + 2 + \cdots + k)x^2 + a_3 x^3 + \cdots\right)$$

for some constants $a_0, a_1, a_3, \ldots$. It follows that the x^2 coefficient of $(1 + x)^{k+2}$ is $(1 + 2 + \cdots + k) + a_1$. Unfortunately, without knowing a_1 we cannot carry through on the inductive step.

This is a situation that occurs quite often in inductive arguments. The initial propositions $P(1), P(2), \ldots$ do not carry enough information to enable one to complete the inductive step. When this happens, it is natural to reformulate the propositions into a stronger, more general form, $Q(1)$, $Q(2), \ldots$ (so that $Q(n)$ implies $P(n)$ for each n), and to look once again for an inductive proof.

So, let $Q(n)$ denote the statement that the coefficients of x^0, x^1, x^2 in the expansion of $(1 + x)^{n+1}$ are 1, $n + 1$, and $1 + 2 + \cdots + n$, respectively. Note that $Q(n)$ implies $P(n)$ for all $n \geqslant 1$.

It easy to check that $Q(1)$ is true. Assume that $Q(k)$ is true. Then using the inductive assumption,

$$\begin{aligned}(1 + x)^{k+2} &= (1 + x)(1 + x)^{k+1}\\ &= (1 + x)\left[1 + (k + 1)x + (1 + 2 + \cdots + k)x^2 + \cdots\right]\\ &= 1 + (k + 2)x + \left(1 + 2 + \cdots + (k + 1)\right)x^2 + \cdots .\end{aligned}$$

This last expression shows that $Q(k + 1)$ is true; so by induction, $Q(n)$ holds for all $n \geqslant 1$. Therefore, $P(n)$ is true for $n \geqslant 1$.

By the binomial theorem, the x^2 coefficient of $(1 + x)^{n+1}$ is $\binom{n+1}{2}$. This, together with the previous result, shows that for $n \geqslant 1$:

$$1 + 2 + \cdots + n = \binom{n+1}{2} = \frac{(n + 1)n}{2} .$$

Exercise. Use mathematical induction to show that the coefficient of x^2 in the expansion of $(1 + x + x^2 + \cdots + x^n)^n$ is $1 + 2 + \cdots + n$. [Note: A counting argument (see Section 4) shows that the x^2 coefficient is $\binom{n}{1} + \binom{n}{2}$.]

4. Counting Arguments. The analysis of algorithms requires a background in the theory and practice of enumerative combinatorics (e.g., combinations and permutations). It is often the case that a counting problem can be done in several different ways, and this provides an interesting technique for proving arithmetic identities.

Consider, for example, an $n \times n$ square grid of lattice points, shown in Figure 6 for $n = 6$. The lattice points can be identified with ordered pairs (i, j), where i $(1 \leqslant i \leqslant n)$ is the column number and j $(1 \leqslant j \leqslant n)$ is the row number.

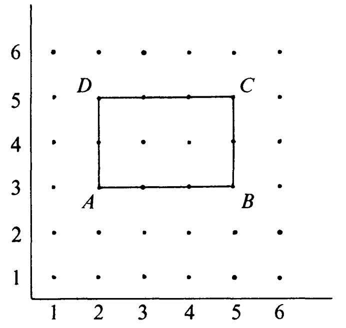


Figure 6.

How many rectangles, with vertical and horizontal sides (such as $ABCD$), can be formed from these lattice points?

One way to proceed is to fix a lattice point, say (i, j), and count the number of rectangles having lower-left corner at vertex (i, j). There are $n - i$ choices for the horizontal dimension and $n - j$ choices for the vertical dimension, and therefore there are $(n - i)(n - j)$ such rectangles. It follows that the total number of rectangles in the $n \times n$ square grid is

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (\text{Number of rectangles having lower-left corner at vertex } (i, j))$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} (n - i)(n - j)$$

$$= \sum_{i=1}^{n} (n - i) \sum_{j=1}^{n} (n - j)$$

$$= \left[1 + 2 + \cdots + (n - 1)\right]^2.$$

On the other hand, a rectangle on the $n \times n$ grid is uniquely determined by two rows and two columns (i.e., the sides of the rectangle). There are $\binom{n}{2}$ ways to choose the columns and $\binom{n}{2}$ ways to choose the rows, and therefore the number of rectangles is $\binom{n}{2}^2$.

Equating this answer to that obtained earlier, we get

$$1+2+\cdots+(n-1)=\binom{n}{2}=\frac{n(n-1)}{2}.$$

Exercise. On the $n \times n$ grid, let S_n denote the number of rectangles whose lower-left corner is either on the first row or the first column (or both).

(a) Show that $S_n = (n-1)^3$.
(b) Show that the $n \times n$ grid has precisely $S_n + S_{n-1} + \cdots + S_1$ rectangles.
(c) Use these results together with the results of this section to show that

$$1^3+2^3+\cdots+n^3=(1+2+\cdots+n)^2=\left(\frac{n(n+1)}{2}\right)^2.$$

5. One-to-One Correspondence. One of the most recent threads of research in combinatorics is to give "bijective" proofs of combinatorial identities. The idea is to set up a one-to-one correspondence (bijection) between two sets, where the sets are carefully chosen so that the left and right sides of the identity count the *number* of elements in the respective sets.

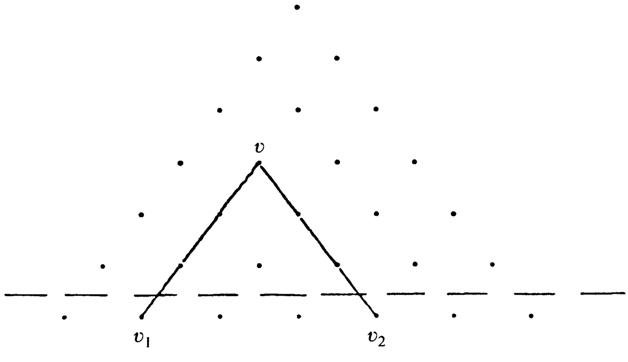


Figure 7.

In Figure 7, we see a natural correspondence between the *vertices* in the top n rows and the *pairs of vertices* (unordered pairs) in the $(n+1)$st row of the triangular array. Specifically, suppose v is a vertex from among the top n rows. From v we can reach the $(n+1)$st row by following a left path (take the left branch at each juncture) or by following a right path (take the right path at each juncture). These two paths terminate at two vertices v_1 and v_2 in the $(n+1)$st row. Let v correspond to the pair $\{v_1, v_2\}$.

Conversely, if $\{v_1, v_2\}$ is any unordered pair of vertices in the $(n+1)$st row, there is a unique vertex v among the first n rows that will correspond to the pair $\{v_1, v_2\}$ in the way described in the previous paragraph.

There are $1+2+\cdots+n$ vertices in the top n rows and $\binom{n+1}{2}$ pairs in the $(n+1)$st row, and therefore

$$1+2+\cdots+n=\binom{n+1}{2}=\frac{(n+1)n}{2}.$$

Exercise. (a) Show that there is a one-to-one correspondence between the set of ordered pairs of integers $\{(a,b): 1 \leqslant a \leqslant b \leqslant n\}$ and the set of ordered pairs of integers $\{(a,b): 1 \leqslant a < b \leqslant n+1\}$. The latter set has $\binom{n+1}{2}$ elements and the former set has $1+2+\cdots+n$ elements.

(b) Show that there is a one-to-one correspondence between the set of ordered triples of integers $\{(a,b,c): 1 \leqslant a \leqslant b \leqslant c \leqslant n\}$ and the set of ordered triples of integers $\{(a,b,c): 1 \leqslant a < b < c \leqslant n+2\}$. What arithmetical identity corresponds to this correspondence?

6. Recurrence Relations. Recursive algorithms, those procedures designed to invoke themselves, form an important class of algorithms in computer science. An analysis of such algorithms leads, in a natural way, to problems involving recurrence relations.

Consider the sequence of real numbers $\{x_n : n = 1, 2, \ldots\}$ defined recursively by the formula

$$nx_n = (n-2)x_{n-1} + 1, \qquad x_1 = 1.$$

The first few terms of the sequence are readily computed,

$$1, 1/2, 1/2, 1/2, \ldots,$$

and it is easy to show that the sequence remains constant thereafter, since $x_{n-1} = 1/2$ implies that $nx_n = (n-2)(1/2) + 1 = n/2$.

There is another way to look at the recurrence relation. Multiply each side of the recurrence by $n-1$ to get

$$n(n-1)x_n = (n-1)(n-2)x_{n-1} + (n-1),$$

and now let $y_n = n(n-1)x_n$. With this substitution the above equation becomes

$$y_n = y_{n-1} + (n-1),$$

or equivalently,

$$y_n - y_{n-1} = n-1.$$

Since this holds for all n, it follows that

$$(y_{n+1}-y_n)+(y_n-y_{n-1})+\cdots+(y_2-y_1) = n+(n-1)+(n-2)+\cdots+1,$$

or equivalently,

$$1+2+\cdots+n = y_{n+1}-y_1.$$

But $y_{n+1} = (n+1)nx_{n+1}$ (so, $y_1 = 0$) and since we know that $x_n = 1/2$ for $n \geqslant 2$, it follows that

$$1 + 2 + \cdots + n = (n+1)nx_n = \frac{n(n+1)}{2}.$$

Exercise. Given the 2×2 matrix

$$M_n = \begin{pmatrix} 1 + 2 + \cdots + n & n \\ n+1 & 2 \end{pmatrix},$$

subtract the second column from the first and take the transpose. The result is the matrix M_{n-1}. Since neither of these operations changes the value of the determinant, it must be the case that $\det M_n = \det M_{n-1}$. Use this recurrence to evaluate $\det M_n$, and use this result to give another proof of the arithmetic sum formula.

Exercise. Find a recurrence relation for the sequence $\{x_n : n = 1, 2, 3, \ldots\}$ defined by $x_n = n(n+1)/2$ (i.e., express x_n in terms of x_{n-1}). Use this recurrence to show that $x_n = 1 + 2 + \cdots + n$.

7. Generating Functions. The generating function for a sequence of numbers

$$a_0, a_1, a_2, a_3, \ldots$$

is defined to be the series

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots.$$

For example, the generating function for the constant sequence

$$1, 1, 1, 1, 1, \ldots$$

is the infinite geometric series

$$1 + x + x^2 + x^3 + x^4 + \cdots,$$

which we recognize as the power series of the function $1/(1-x)$ when $|x| < 1$.

This geometric series has a very important property as far as generating functions are concerned. Its product with another generating function,

$$(1 + x + x^2 + x^3 + \cdots)(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots)$$

yields the series

$$a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + \cdots + (a_0 + a_1 + \cdots + a_n)x^n + \cdots.$$

Notice that the coefficients in the product are partial sums of the coefficients of the original series. Therefore, we see that

$$\begin{aligned} \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x}\right) &= (1 + x + x^2 + x^3 + \cdots)(1 + x + x^2 + x^3 + \cdots) \\ &= 1 + (1+1)x + (1+1+1)x^2 + \cdots \\ &= 1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdots \end{aligned}$$

and

$$\left(\frac{1}{1-x}\right)\left(\frac{1}{1-x}\right)^2 = (1 + x + x^2 + x^3 + \cdots)(1 + 2x + 3x^2 + 4x^3 + \cdots)$$

$$= 1 + (1+2)x + (1+2+3)x^3 + \cdots$$

$$+ (1 + 2 + \cdots + n)x^{n-1} + \cdots. \tag{1}$$

This identity shows that the coefficient of x^{n-1} in the power series expansion of $1/(1-x)^3$ is $1 + 2 + \cdots + n$.

Another way to find the power series of $1/(1-x)^3$ is to consider $|x| < 1$ and take two derivatives of the power series of $1/(1-x)$. Thus, starting from

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^{n+1} + \cdots,$$

one derivative yields

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots + (n+1)x^n + \cdots,$$

and a second derivative gives

$$\frac{2}{(1-x)^3} = 2 + 6x + \cdots + n(n+1)x^{n-1} + \cdots.$$

It follows that

$$\frac{1}{(1-x)^3} = 1 + 3x + \cdots + \frac{n(n+1)}{2}x^{n-1} + \cdots. \tag{2}$$

Since the power series representation of a function is unique, the coefficients of x^{n-1} in (1) and (2) must be equal; that is,

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Exercise. Compute the power series of $\dfrac{1}{ax^2 - (a+1)x + 1}$, $a \neq 1$, in two different ways, and discover the formula for the sum of the finite geometric series by equating the coefficients of x^n. One way to compute the power series is to begin by writing $\dfrac{1}{ax^2 - (a+1)x + 1} = \left(\dfrac{1}{1-ax}\right)\left(\dfrac{1}{1-x}\right)$; another way is to begin with the partial fraction decomposition,

$$\frac{1}{ax^2 - (a+1)x + 1} = \frac{1}{1-a}\left[\left(\frac{1}{1-x}\right) - \left(\frac{a}{1-ax}\right)\right].$$

8. Calculus. The preceding section shows that calculus is a useful tool in conjunction with discrete mathematics. For another example, consider the identity

$$(1-x)(1+x+x^2+\cdots+x^n)=1-x^{n+1}.$$

Take the derivative of each side to get

$$-(1+x+x^2+\cdots+x^n)+(1-x)(1+2x+\cdots+nx^{n-1})=-(n+1)x^n,$$

and now take the derivative of each side of this to obtain

$$\begin{aligned}-(1+2x+\cdots+nx^{n-1})-(1+2x+\cdots+nx^{n-1})&\\ +(1-x)\big(2+\cdots+n(n-1)x^{n-2}\big)&\\ =-n(n+1)x^{n-1}.&\end{aligned}$$

Setting $x=1$ yields the arithmetic formula

$$1+2+\cdots+n=\frac{n(n+1)}{2}.$$

Exercise. Find formulas for the following finite series.

(a) $$\binom{n}{1}+2\binom{n}{2}+3\binom{n}{3}+\cdots+n\binom{n}{n}.$$

(b) $$1+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+\cdots+\frac{1}{n+1}\binom{n}{n}.$$

[Hint: Begin with $\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\cdots+\binom{n}{n}x^n=(1+x)^n$.]

9. Finite Differences. The derivative of a function $y=f(x)$ at x is defined to be

$$Df(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h},$$

whenever this limit exists. D is the differentiation operator which, when applied to a function, produces the derivative of that function.

Discrete mathematics has a similar operator that can be applied to a function. The *first difference* of the function f at $x=n$ is defined to be

$$\Delta f(n)=f(n+1)-f(n).$$

The symbol Δ denotes the difference operator. If a function f defines the sequence

$$f(0), f(1), f(2), f(3), \ldots, f(n), \ldots$$

then the function Δf defines the sequence

$$\Delta f(0), \Delta f(1), \Delta f(2), \ldots, \Delta f(n), \ldots .$$

The difference sequence leads to a calculus of finite differences that is completely analogous to the ordinary calculus of differentiable functions. Some of the similarities are shown in the table below, where $x^{(k)} = x(x-1)(x-2)\cdots(x-(k-1))$ for positive integers k.

Differential Calculus	Difference Calculus
1. $D(cf) = cD(f)$	1′. $\Delta(cf) = c(\Delta f)$
2. $D(af+bg) = aDf + bDg$	2′. $\Delta(af+bg) = a(\Delta f) + b(\Delta g)$
3. $D\left(\dfrac{f}{g}\right) = \dfrac{gDf - fDg}{g^2}$	3′. $\Delta\left(\dfrac{f}{g}\right) = \dfrac{g\Delta f - f\Delta g}{g^2}$
4. $Dx^n = nx^{n-1}$	4′. $\Delta x^{(n)} = nx^{(n-1)}$
5. $\int_1^n x^k\,dx = \dfrac{n^{k+1}-1}{k+1}$	5′. $\sum_{x=1}^{n} x^{(k)} = \dfrac{(n+1)^{(k+1)} - 1^{(k+1)}}{k+1}$.

One can readily verify (5′) using (∗) on page 372. When $k=1$, formula (5′) yields

$$\begin{aligned} 1+2+\cdots+n &= \frac{(n+1)^{(2)} - 1^{(2)}}{2} \\ &= \frac{(n+1)n - 1(1-1)}{2} \\ &= \frac{n(n+1)}{2}. \end{aligned}$$

There are also analogues for higher derivatives. Just as $D^2 f = D(Df)$ and $D^n f = D(D^{n-1}f)$, we can define $\Delta^n f = \Delta(\Delta^{n-1} f)$ for each n. The formula which corresponds to the Taylor series

$$f(x) = \sum_{k=0}^{\infty} \frac{D^k f(a)}{k!}(x-a)^k \tag{6}$$

is

$$f(n) = \sum_{k=0}^{\infty} \frac{\Delta^k f(a)}{k!}(x-a)^{(k)} \qquad \text{(positive integer } a\text{).} \tag{6′}$$

This follows from Theorem 2 of [2] by noting that $\dfrac{(n-a)^{(k)}}{k!} = \dbinom{n-a}{k}$.

As an example, suppose that f is defined by

$$f(0) = 0 \quad \text{and} \quad f(n) = 1 + 2 + \cdots + n$$

for $n = 1, 2, 3, \ldots$. We will use the Maclaurin series analog of (6)

$$f(n) = \sum_{k=0}^{\infty} \frac{\Delta^k f(0)}{k!} n^{(k)} \tag{7′}$$

to find a closed-form expression for $f(n)$. The following table gives the values of $f, \Delta f, \Delta^2 f, \ldots$ at $n = 0, 1, 2, 3, \ldots$.

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	
f:	0	1	3	6	10	15	$\cdots$
Δf:	1	2	3	4	5	6	$\cdots$
$\Delta^2 f$:	1	1	1	1	1	1	$\cdots$
$\Delta^3 f$:	0	0	0	0	0	0	$\cdots$

All successive rows are identically equal to zero. By (7′),

$$1 + 2 + \cdots + n = \frac{f(0)}{0!} n^{(0)} + \frac{\Delta f(0)}{1!} n^{(1)} + \frac{\Delta^2 f(0)}{2!} n^{(2)}$$

$$= 0 + n + \frac{1}{2!} n(n-1)$$

$$= \frac{n(n+1)}{2}.$$

Exercise. Use (7′) to find a closed-form expression for the sum of the first n cubes: $f(0) = 0$ and $f(n) = 1^3 + 2^3 + \cdots + n^3$.

10. Constructive Combinatorics. We have considered a number of mathematical ideas that are important in discrete mathematics: iteration, induction, recursion, elementary enumeration, generating functions, finite differences. Certainly these ideas are not new to mathematics; indeed, they have been part of the mainstream since the time of Euler. But they are of growing importance today because of their applicability to problems which arise within the context of computing and computer science. It is the finite nature of the computer which leads us to think in discrete terms.

A final example will illustrate the constructive and algorithmic flavor characteristic of much of modern discrete mathematics. Take a deck of $T_n = n(n+1)/2$ cards and divide it into an arbitrary number of piles. Now form a new pile by taking one card from each of the piles. Continue to repeat this process: remove a card from each pile and form a new pile. Surprisingly, the piles will eventually converge to n piles of decreasing size, with n cards in the largest pile, $n - 1$ in the next largest, and so forth!

As an example, suppose we divide a deck of $T_6 = 21$ cards into three piles of 5, 9, and 7 cards. We can arrange the piles in any order, say, from left to right, and represent this situation by the triple $(5, 9, 7)$. After removing one card from each of these piles, we obtain four piles represented by $(3, 4, 8, 6)$. Here, we have placed the newly created pile on the far left and the remaining piles remain in their same relative order but with $4, 8, 6$ cards respectively. We will symbolize this step by writing $(5, 9, 7) \to (3, 4, 8, 6)$.

If we continue to create new piles in this way, we will produce the following sequence: $(5,9,7) \to (3,4,8,6) \to (4,2,3,7,5) \to (5,3,1,2,6,4)$. In this case, we arrive at six piles of different sizes in just three steps. (It usually takes more; for instance, 20 steps are required when we begin with $(3,5,6,7)$.) It is interesting to note that continued repetition of the steps of the algorithm will produce a sorted arrangement (with our notation): $(5,3,1,2,6,4) \to (6,4,2,1,5,3) \to (6,5,3,1,4,2) \to (6,5,4,2,3,1) \to (6,5,4,3,1,2) \to (6,5,4,3,2,1)$.

The proof that this procedure always terminates in this way is rather complicated to express, but it is based on the kinds of thinking discussed in this paper: search for patterns, induction, recursion. (See [1] for details.) It is important to point out that the computer can be a useful tool in investigating this situation. Given a partition of the deck (an input), a simple program can be written to print out all the intermediate arrangements leading to the final limiting pattern. This is bound to make it easier for us to notice patterns and make conjectures. It may even suggest further questions. For example, what is the maximum number (or average number) of steps before termination? Is there a way of predicting the number of steps for a given arrangement? How many different cuts require exactly k steps? Playing with these questions for various configurations will lead to conjectures, and more questions, and more conjectures, and so forth. As this example shows, even beginning students in discrete mathematics can be actively involved in the creative process.

Exercise. The algorithm of this section again illustrates that $1 + 2 + \cdots + n = n(n+1)/2$. Modify the algorithm to show that

$$1 + 3 + 5 + \cdots + (2n-1) = n^2.$$

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